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Revisiting the gauge principle: enforcing constants of motion as constraints

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Abstract

In this paper we examine an alternative formulation of the gauge principle in which the emphasis is shifted from the symmetry transformations to their generators. We show that the gauge principle can be entirely reformulated in terms of promoting constants of motion—which generate rigid symmetries—to constraints—which generate gauge symmetries. In our exposition we first explain the basic philosophy on mechanical systems, and then with the help of De Donder–Weyl formalism we extend our scenario also to a field-theoretical setting. To illustrate this, we demonstrate our method in numerous examples, including the massive relativistic particle, the Nambu–Goto closed string and relativistic field theory.

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1. Introduction

The gauge principle (see [1] for a historical account) is a basic ingredient of modern theoretical physics, particularly in quantum field theory. It is not necessary to elaborate much on this undisputable fact. A quick presentation of its main idea is that by gauging a rigid symmetry one must pay a 'price': that of introducing a new field, the gauge field, which geometrically represents a principal connection on a principal bundle. This 'price' has turned out to be an unexpected bonus which has irrevocably changed the theoretical landscape in physics.

In this paper we propose to revisit the gauge principle from the point of view of enforcing constants of motion as constraints. We should, however, forewarn that our subsequent considerations will be purely classical, so particularly, ordering issues will be outside our scope. Similarly we will assume that Lagrangian/Hamiltonian systems are equivalent when they produce identical equations of motion (EOM). This 'on mass-shell' (i.e. the classical path) identification is clearly not satisfactory from a quantum point of view where also 'off mass-shell' behavior non-trivially contributes into, say, transitional amplitudes. Grassmann variables will also not be considered, since that complication is a straightforward generalization.

It is well known that theories—derived form a variational principle—which exhibit gauge invariance must be described by constrained systems. With these two words we refer to the framework put forward by Rosenfeld [2, 3], Dirac [4, 5] and Bergmann [6–8], who, independently, laid the ground to deal with such systems. In particular Rosenfeld's contribution, which has been overlooked for a long time, has recently resurfaced thanks to the work of Salisbury and it is discussed in [2]. The constrained systems are characterized by Lagrangians whose Hessian matrix with respect to the velocities is singular, thus preventing the Legendre map (LM) from tangent bundle (i.e. positions and velocities space) to cotangent (i.e. space of positions and momenta, or phase space) from being invertible. It is precisely the singularity of the Hessian matrix which makes room for the possible presence of gauge freedom. Eventually, the picture obtained in phase space is that we have a (non-uniquely defined) canonical Hamiltonian H, and a set of primary constraints C_a that are just the consequence of the non-invertibility of the LM. Thus the dynamics in phase space is given by the Dirac Hamiltonian

$$H_D := H + \lambda^a C_a, \tag{1}$$

with λ^a being a set of in principle arbitrary Lagrange multipliers, together with the requirement that motions must satisfy the primary constraints

$$C_a = 0. \tag{2}$$

Here we will not mention the details of the theory of constrained systems, but simply refer to the literature [9–12]. What we want to emphasize is that, given the structure of the dynamics in phase space, one could think of a process of gauging a regular theory by just starting with an ordinary Hamiltonian H and a set of functions C_a that are to be enforced as constraints. Then we could define a new dynamics by equations (1) and (2), which hopefully would describe a gauge theory.

In general, this program is bound to fail because the constraints must have a certain degree of compatibility with the generator H_D of the dynamics. Geometrically, one needs the dynamical trajectories to be tangent to the surface defined by the constraints. In general, one expects this requirement to eventually end up with the appearance of new constraints as well as the determination of some of the Lagrange multipliers. But if H and C_a are chosen too arbitrarily, the most likely outcome is that there will be no set of λ^a's that keeps the dynamical trajectories tangent to the constraint's surface.

But there is a neat exception, with plenty of interest: if we choose the would-be constraints as some of the constants of motion for H, then full compatibility is easy to achieve. This is the case we will explore. We consider a Hamiltonian for a regular theory—obtained from a Lagrangian in tangent space through an invertible LM—and a set of constants of motion C_a satisfying $\{C_a, H\} = 0$ and closing a certain algebra $\{C_a, C_b\} = c_{ab}^c C_c$ with c_{ab}^c being structure constants. For the sake of simplicity, we restrict ourselves to constants of motion without explicit time dependence, i.e. to scleronomic constants of motion. We then declare that the new dynamics is governed by the Dirac Hamiltonian (1) under the condition that the constants of motion are enforced now as constraints (2).

To check that we are indeed on the right track, we must verify that with these conditions the theory defined by (1) supports gauge symmetries and that they act on the 'matter' fields as they should, just generalizing the action of the former rigid symmetries. Once this check is done, we can explore the new gauge theory and its dynamical consequences, because the dynamics is expected to undergo important changes after the gauging of the rigid group of symmetries. Finally we can further modify the theory in a natural way by introducing gauge-invariant kinetic terms for the Lagrange multipliers. The full-fledged gauge theory is then obtained, with the new nontrivial interaction terms allowed by the gauge principle.

Our paper is organized as follows. In section 2 we formulate our basic strategy using the language of mechanics. Namely, we show how to construct a gauge-invariant theory by promoting constants of motion to constraints. We also stress an intimate connection with the mathematical structure of non-Abelian Yang–Mills theory [13]. In section 3, we complete the theoretical setup. Examples in mechanics are given in section 4 and the relativistic field theory is dealt with in section 5, where the key role of the De Donder–Weyl formalism is made manifest. We devote section 6 to the case of the closed bosonic string and use our approach to obtain world sheet general covariance. Finally, we conclude in section 7 with a brief summary of our results and outlook.

2. The new gauge theory

We start by considering a Hamiltonian for a regular theory together with a set of scleronomic constants of motion C_a satisfying $\{C_a, H\} = 0$ and closing an algebra $\{C_a, C_b\} = c^c_{ab} C_c$. Now we will prove that when C_a are enforced now as constraints, then this new theory is indeed a gauge theory. The simplest way to prove it is by defining the extended Lagrangian (indices for vector components are normally suppressed)

$$L_{\mathrm{e}}(q, p, \dot{q}, \dot{p}, \lambda) = p\dot{q} - H(q, p) - \lambda^a C_a(q, p), \tag{3}$$

and proving that it has gauge transformations. Note first that the EOM for (3) coincide with those derived from the Dirac Hamiltonian (1) and the constraints (2) (an advantage of the Lagrangian formulation is that all the dynamics is encoded in a single function). Note also that we have enlarged the configuration space with the multipliers λ^a as new variables. We will prove that indeed (3) has Noether gauge symmetries. Since the constants of motion are the Noether generators of the rigid symmetries, it is reasonable to expect that the generator of the would-be canonical gauge transformations can be written as $G \equiv \epsilon^a(t) C_a$, with ϵ^a being a set of arbitrary time-dependent functions. We will prove now that indeed G generates gauge transformations. The corresponding variations can be written as

$$\delta_\epsilon q^i = \{q^i, G\}, \qquad \delta_\epsilon p_i = \{p_i, G\}, \tag{4}$$

and the variations of the multipliers will be determined below by the condition that, under the variations thus defined, the Lagrangian L_{e} is quasi-invariant, i.e.

$$\delta_\epsilon L_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} F, \tag{5}$$

for some F linear in ϵ and its derivatives. Indeed,

$$\begin{aligned}
\delta_\epsilon (p_i \dot{q}^i) &= p_i \delta_\epsilon \dot{q}^i + \dot{q}^i \delta_\epsilon p_i = -\dot{p}_i \delta_\epsilon q^i + \dot{q}^i \delta_\epsilon p_i + \frac{\mathrm{d}}{\mathrm{d}t}(p_i \delta_\epsilon q^i) \\
&= -\epsilon^a(t) \frac{\partial C_a}{\partial p_i} \dot{p}_i - \epsilon^a(t) \frac{\partial C_a}{\partial q^i} \dot{q}^i + \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i \epsilon^a(t) \frac{\partial C_a}{\partial p_i} \right) \\
&= \dot{\epsilon}^a(t) C_a + \frac{\mathrm{d}}{\mathrm{d}t}\left[\epsilon^a(t) \left(p_i \frac{\partial C_a}{\partial p_i} - C_a \right) \right]
\end{aligned} \tag{6}$$

and

$$\delta_\epsilon (H + \lambda^a C_a) = \epsilon^b(t) \{H, C_b\} + C_a \delta_\epsilon \lambda^a + \lambda^a \epsilon^b(t) \{C_a, C_b\} = C_a \delta_\epsilon \lambda^a + \lambda^a \epsilon^b(t) c^c_{ab} C_c. \tag{7}$$

Thus, the appropriate definition

$$\delta_\epsilon \lambda^a := \dot{\epsilon}^a(t) - \lambda^b \epsilon^c(t) c^a_{bc} =: (D_0 \epsilon(t))^a \tag{8}$$

makes $\delta_\epsilon L_{\mathrm{e}}$ to be

$$\begin{aligned}\delta_\epsilon L_{\mathrm{e}} &= C_a\big(\dot{\epsilon}^a(t) - \delta_\epsilon \lambda^a - \lambda^b \epsilon^c(t) c^a_{bc}\big) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\epsilon^a(t)\left(p_i \frac{\partial C_a}{\partial p_i} - C_a\right)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\epsilon^a(t)\left(p_i \frac{\partial C_a}{\partial p_i} - C_a\right)\right],\end{aligned} \tag{9}$$

which proves that variations (4) and (8) define a Noether gauge symmetry for L_{e}. In equation (8) we have introduced the covariant derivative [14]

$$(D_0)^a_c := \partial_t \delta^a_c - \lambda^b c^a_{bc}, \tag{10}$$

which is nothing but the covariant derivative for the adjoint representation. Analogously, one can introduce the covariant derivative for the phase-space variables $\xi^i = \{p_1, \ldots, q^1, \ldots\}$ as

$$D_0 \xi^i := \partial_t \xi^i - \lambda^a \Gamma_a \xi^i = \partial_t \xi^i - \lambda^a \{\xi^i, C_a\}. \tag{11}$$

Here Γ_a is the representation of the symmetry generators C_a that acts on ξ^i. Indeed, the Jacobi identity for Poisson brackets ensures that $[\Gamma_a, \Gamma_b] = -c^c_{ab}\Gamma_c$. Because both ∂_t and $\{, C_a\}$ fulfill the Leibniz rule one can extend the covariant derivative (11) to any function $\phi(\boldsymbol{\xi})$ on the phase space.

Using the fact that our active variations commute with the time derivatives, i.e. $\delta_\epsilon(\partial_t \phi) = \partial_t(\delta_\epsilon \phi)$, it is easy to check that the covariance condition takes the form

$$\delta_\epsilon(D_0 \phi) = \epsilon^a D_0(\{\phi, C_a\}) =: \epsilon^a \Gamma_a(D_0 \phi). \tag{12}$$

The first equality in (12) can be proved by considering (1) that active variations commute with the time derivatives, and so $\Gamma_a(\partial_t \phi) = \partial_t\{\phi, C_a\}$, and (2) that the action of Γ_a on the multipliers is the adjoint action: $\Gamma_a \lambda^b = c^b_{ac}\lambda^c$. Note that in the last equality in (12), the representation Γ_a of the symmetry generators acting on $D_0 q$ was defined. This definition turns out to be an exact identity on the mass-shell. In this regard it is interesting to realize that the curvature $\mathbf{F} = [D_0, D_0] = 0$, and so in the case of mechanics the usual gauge-invariant kinetic term $\mathrm{Tr}(\mathbf{F}^2)$ is trivially zero. Thus the multipliers λ^a cannot become dynamical variables. On the other hand, a subsequent elimination of the momenta—which are auxiliary variables (auxiliary variables are by definition variables that can be isolated by using their own EOM) for L_{e}—as done in the next subsection—will assign the λ^a's the status of auxiliary variables.

Note that (8), (10) and (11) carry indeed all the flavor of the transformation of a gauge field in a non-Abelian gauge theory. This is exactly the case, because what we have done is precisely the application of the gauge principle: to gauge a group of rigid symmetries. We remind that the rigid symmetries are generated by constants of motion while the gauge symmetries by the first-class constraints[4]. Thus gauging a group of rigid symmetries is tantamount to enforce the generating constants of motion as constraints. In this respect λ^a play the role of a connection in a principal bundle over $\mathbb{R}$. The fact that $\mathbf{F} = 0$ then indicates that this bundle is flat (not a big surprise for a bundle with so simple base space). Let us, however, stress that our derivation would go through even if not every rigid symmetry is gauged. For instance, we could have limited ourselves only to gauging any subgroup of rigid symmetries. The analogy with non-Abelian Yang–Mills theory is summarized in table 1.

[4] Dirac introduced the concept of a first-class function as a function whose Poisson bracket with the constraints vanishes on the constraints' surface. Gauge symmetries are made up with linear combinations of first-class constraints, which appear as coefficients of an expansion in terms of the arbitrary functions and their time derivatives up to a certain order.

Table 1. Comparison between the gauge theory presented in section 2 and the non-Abelian Yang–Mills theory. The parallelism obtained allows us to formally identify $D_0 \leftrightarrow D_\mu$ and $\lambda \leftrightarrow \boldsymbol{A}_\mu$.

Gauge theory from section 2[a]	Non-Abelian Yang–Mills theory[b]
$\delta_\epsilon \lambda^a(t) = D_0 \epsilon^a(t)$	$\delta A^a_\mu(x) = D_\mu \epsilon^a(x)$
$\delta_\epsilon \phi(\xi) = \Gamma(\boldsymbol{\epsilon})\phi(\xi)$	$\delta \boldsymbol{\Phi}(x) = iT(\boldsymbol{\epsilon})\boldsymbol{\Phi}(x)$
$D_0 \epsilon^a(t) = \partial_t \epsilon^a(t) - \lambda^b(t) c^a_{bc} \epsilon^c(t)$	$D_\mu \epsilon^a(x) = \partial_\mu \epsilon^a(x) + A^b_\mu(x) f^a_{bc} \epsilon^c(x)$
$D_0 \phi(\xi) = \partial_t \phi(\xi) - \lambda^a \Gamma_a \phi(\xi)$	$D_\mu \boldsymbol{\Phi}(x) = \partial_\mu \boldsymbol{\Phi}(x) - iA^a_\mu T_a \boldsymbol{\Phi}(x)$
$\{C_a, C_b\} = c^c_{ab} C_c \Rightarrow [\Gamma(C_a), \Gamma(C_b)] = -c^a_{ab}\Gamma(C_a)$	$[t_a, t_b] = \mathrm{i} f^c_{ab} t_c \Rightarrow [T(t_a), T(t_b)] = \mathrm{i} f^c_{ab} T(t_c)$
On mass-shell situation ($\partial_t C_a = 0$)	
$D_0 \boldsymbol{\epsilon} = \partial_t \boldsymbol{\epsilon} - \{\boldsymbol{\lambda}, \boldsymbol{\epsilon}\}$	$D_\mu \boldsymbol{\epsilon} = \partial_\mu \boldsymbol{\epsilon} + [\boldsymbol{A}_\mu, \boldsymbol{\epsilon}]$
$\delta_\epsilon(D_0\phi) = D_0(\{\phi, \boldsymbol{\epsilon}\}) = \Gamma(\boldsymbol{\epsilon}) D_0 \phi$	$\delta(D_\mu \boldsymbol{\Phi}) = \mathrm{i}T(\boldsymbol{\epsilon}) D_\mu \boldsymbol{\Phi}$

[a] Here we accept notations: $\boldsymbol{\lambda} = \lambda^a C_a$, $\boldsymbol{\epsilon} = \epsilon^a C_a$, $\xi = \{p_1, p_2, \ldots, q^1, q^2, \ldots\}$ is a phase-space point, ϕ is an arbitrary function on a phase space and $\Gamma(C_a) = \Gamma_a = \{, C_a\} = \omega^{ij} \frac{\partial C_a}{\partial \xi^j} \frac{\partial}{\partial \xi^i}$.

[b] Here we accept notations: $\boldsymbol{A}_\mu = -\mathrm{i} A^a_\mu t_a$, $\boldsymbol{\epsilon} = \epsilon^a t_a$, $\boldsymbol{\Phi}$ is an arbitrary field multiplet and $T(t_a) = T_a$ is an irreducible representation of the algebra of t_a generators that is adapted to $\boldsymbol{\Phi}$, e.g. for $\boldsymbol{\Phi}$ in fundamental representation of $SU(N)$ then $T(\boldsymbol{A}_\mu) = -\mathrm{i} A^a_\mu T_a$ with T_a being the $(N \times N)$ Hermitian matrices. Generators in self-adjoint rep. are defined as $(T_b)^a_c = \mathrm{i} f^a_{bc}$.

Note finally that the case of a soft algebra [10] is easily accommodated. We can relax the condition that the constants of motion C_a form a Lie algebra to that of a soft algebra, where there are no longer structure constants but structure functions, $\{C_a, C_b\} = c^c_{ab}(q, p) C_c$, and we can also relax the constant of motion condition, $\{C_a, H\} = 0$ to $\{C_a, H\} = a^b_a(q, p) C_b$. In this case, equation (8) changes to

$$\delta_\epsilon \lambda^a := \dot{\epsilon}^a(t) - \lambda^b \epsilon^c(t) c^a_{bc} - \epsilon^b(t) a^b_a. \tag{13}$$

In the field theory, one can find more general cases [10], like that of an open algebra, where the algebra of the constants of motion only closes up to linear terms that are antisymmetric combinations of the equations of motion, or when there is functional dependence among the constants of motion. We believe that these cases can also be addressed, but since the ordinary case already requires a non-standard formalism (see subsection 5.3), we leave them for further study.

3. Inverting the Legendre map

3.1. The Lagrangian L_λ

Consider now the equations of motion for L_e, i.e.

$$\dot{q} - \frac{\partial H}{\partial p} - \lambda^a \frac{\partial C_a}{\partial p} = 0, \tag{14a}$$

$$\dot{p} + \frac{\partial H}{\partial q} + \lambda^a \frac{\partial C_a}{\partial q} = 0, \tag{14b}$$

$$C_a = 0. \tag{14c}$$

It is interesting to observe that by introducing the symplectic matrix $\boldsymbol{\omega}$:

$$\omega_{ij} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}_{ij}, \tag{15}$$

(with $\omega_{ij}^{-1} = \omega^{ij}$) the EOM (14) can be succinctly written as

$$D_0\xi^i = \omega^{ij}\frac{\partial H}{\partial \xi^j}, \qquad C_a = 0. \tag{16}$$

We can now use the first set of equations (14) to locally isolate the momenta in terms of positions q, velocities $\dot{q}$ and the multipliers λ^a, thus rewriting (14*a*) in the equivalent form:

$$p - P(q, \dot{q}, \lambda) = 0, \tag{17}$$

for some functions P. This invertibility of the LM will hold in general. In fact, since the starting theory was not gauge, invertibility is guaranteed for $\lambda^a = 0$. With λ^a being just new independent variables, invertibility will be maintained in general.

We implement $p \to P(q, \dot{q}, \lambda)$ into L_e to define the new Lagrangian L_λ:

$$L_\lambda(q, \dot{q}, \lambda) = P(q, \dot{q}, \lambda)\dot{q} - H(q, P(q, \dot{q}, \lambda)) - \lambda^a C_a(q, P(q, \dot{q}, \lambda)). \tag{18}$$

Notice then

$$\begin{aligned}\frac{\partial L_\lambda}{\partial q} &= -\left(\frac{\partial H}{\partial q} + \lambda^a\frac{\partial C_a}{\partial q}\right)\bigg|_{p\to P} + \left(\dot{q} - \frac{\partial H}{\partial p} - \lambda^a\frac{\partial C_a}{\partial p}\right)\bigg|_{p\to P}\frac{\partial P}{\partial \dot{q}} \\ &= -\left(\frac{\partial H}{\partial q} + \lambda^a\frac{\partial C_a}{\partial q}\right)\bigg|_{p\to P},\end{aligned} \tag{19}$$

because $\frac{\partial L_e}{\partial p}\big|_{p\to P} = \left(\dot{q} - \frac{\partial H}{\partial p} - \lambda^a\frac{\partial C_a}{\partial p}\right)\big|_{p\to P}$ vanishes identically owing to the procedure to define the functions $P(q, \dot{q}, \lambda)$. By the same token, we obtain

$$\frac{\partial L_\lambda}{\partial \dot{q}} = P(q, \dot{q}, \lambda), \tag{20}$$

so we reobtain the functions P as the definition of the new Lagrangian momenta. By taking into account equation (20), the reader may note that the EOM for the Lagrangian L_λ yield

$$\frac{\mathrm{d}P}{\mathrm{d}t} + \frac{\partial H}{\partial q} + \lambda^a\frac{\partial C_a}{\partial q} = 0, \tag{21}$$

which is equivalent to (14*b*) when the identity (17) is utilized.

The remaining EOM for L_e is the one associated with the multiplier λ. This equation sets the constraint just as EOM. From the perspective of L_λ, we can write down EOM for λ^a; $\frac{\partial L_\lambda}{\partial \lambda^a} = \frac{\partial L_e}{\partial p}\big|_{p\to P}\frac{\partial P}{\partial \lambda^a} - C_a(q, P(q, \dot{q}, \lambda))$, but again, since $\frac{\partial L_e}{\partial p}\big|_{p\to P}$ vanishes identically, we end up with

$$C_a(q, P(q, \dot{q}, \lambda)) = 0, \tag{22}$$

as the last EOM for L_λ. This shows the equivalence between EOM from L_e and L_λ, because (22) is just (14*c*) with the substitution $p \to P(q, \dot{q}, \lambda)$, which is nothing but the EOM (14*a*).

3.2. Gauge symmetry for L_λ

Let us now prove that L_λ has the gauge symmetry $\delta_\lambda q = (\delta_\epsilon q)|_{p\to P}$, $\delta_\lambda\lambda = \delta_\epsilon\lambda$. One has

$$\delta_\lambda L_\lambda = \left(\frac{\partial L_e}{\partial q}\right)\bigg|_{p\to P}\delta_\lambda q + \left(\frac{\partial L_e}{\partial \dot{q}}\right)\bigg|_{p\to P}\delta_\lambda \dot{q} + \left(\frac{\partial L_e}{\partial p}\right)\bigg|_{p\to P}\delta_\lambda P + \left(\frac{\partial L_e}{\partial \lambda}\right)\bigg|_{p\to P}\delta_\lambda\lambda. \tag{23}$$

We need not care for the term with $\delta_\lambda P$ because $\left(\frac{\partial L_e}{\partial p}\right)\Big|_{p\to P} = 0$ identically due to the equivalence between (14) and (17). As regards $\delta_\lambda \dot{q}$ we can write it as $(\delta_\epsilon \dot{q})|_{p\to P}$. All in all we can write

$$\delta_\lambda L_\lambda = (\delta_\epsilon L_e)|_{p\to P} = \left(\frac{\mathrm{d}}{\mathrm{d}t}F\right)\bigg|_{p\to P} = \frac{\mathrm{d}}{\mathrm{d}t}(F_{|_{p\to P}}), \tag{24}$$

where[5] in the last step we use that $p \to P$ implies also $\dot{p} \to \frac{\mathrm{d}}{\mathrm{d}t}P$, etc. Thus we have proved that L_λ inherits the gauge invariance of L_e.

3.3. A step further

Finally, if λ can be isolated from equation (22), this means that it is in fact an auxiliary variable. It is well known that auxiliary variables can be substituted back into the Lagrangian without affecting the dynamics (see e.g. the appendix in [15]). In fact the earlier substitution $p \to P(q, \dot{q}, \lambda)$ in the previous subsection is an example of this mechanism, for the variables p are isolated by use of their own equations of motion (14), but we have been explicit in the proof of equivalence of EOM. Thus with the substitution we would have arrived at a new Lagrangian $L(q, \dot{q})$ with a dynamics equivalent to that of L_e. Of course, there may be technical obstacles to carrying out this step: solving the system of equations (22) may prove too difficult; getting rid of the multipliers can in general lead to impractically complicated, nonpolynomial expressions for L, etc. One can then revert back to the Lagrangian L_e, with its EOM (14).

3.4. . . . and a step further

Despite potential complications related to solving the system (22) we will suppose that indeed the variables λ^a can be isolated from equations (22) and eliminated by plugging them back into the Lagrangian L_λ. Thus (22) will be equivalent to $\lambda^a = \Lambda^a(q, \dot{q})$ for some functions Λ^a. We will prove that $L(q, \dot{q}) := (L_\lambda(q, \dot{q}, \lambda))|_{\lambda\to\Lambda}$ has the gauge symmetry $\delta_L q = (\delta_\lambda q)|_{\lambda\to\Lambda}$. One has

$$\delta_L L = \left(\frac{\partial L_\lambda}{\partial q}\right)\bigg|_{\lambda\to\Lambda}\delta_L q + \left(\frac{\partial L_\lambda}{\partial \dot{q}}\right)\bigg|_{\lambda\to\Lambda}\delta_L \dot{q} + \left(\frac{\partial L_\lambda}{\partial \lambda}\right)\bigg|_{\lambda\to\Lambda}\delta_L \lambda. \tag{25}$$

Note that we do not have to define $\delta_L \lambda$ because the equation $\lambda = \Lambda(q, \dot{q})$ is exactly $\frac{\partial L_\lambda}{\partial \lambda} = 0$. We continue

$$\delta_L L = \left(\frac{\partial L_\lambda}{\partial q}\delta_\lambda q + \frac{\partial L_\lambda}{\partial \dot{q}}\delta_\lambda \dot{q} + \frac{\partial L_\lambda}{\partial \lambda}\delta_\lambda \lambda\right)\bigg|_{\lambda\to\Lambda} = \left(\frac{\mathrm{d}}{\mathrm{d}t}F_{|_{p\to P}}\right)\bigg|_{\lambda\to\Lambda} = \frac{\mathrm{d}}{\mathrm{d}t}(F_{|_{p\to P,\lambda\to\Lambda}}). \tag{26}$$

This concludes the proof that L is a Lagrangian with gauge symmetry. Our result is general. Given any regular (i.e. non-gauge) theory and a Noether constant of motion in the canonical formalism, one can make this constant of motion a first-class constraint and construct an associated Lagrangian with this gauge symmetry.

[5] Note that the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ means different things depending on the variables we consider. For instance, with variables q, p, λ, we will have $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} + \dot{\lambda}\frac{\partial}{\partial \lambda} + \ddot{q}\frac{\partial}{\partial \dot{q}} + \cdots$. Instead, if we send $p \to P(q, \dot{q}, \lambda)$, we will have $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{q}\frac{\partial}{\partial q} + \dot{\lambda}\frac{\partial}{\partial \lambda} + \ddot{q}\frac{\partial}{\partial \dot{q}} + \cdots$.

4. Examples in mechanics

4.1. Enforcing a function not being a constant of motion as a constraint

Although we are developing the theory for implementing constants of motion as constraints, let us consider an example where one implements a non-constant of motion, just to realize in practical terms the problems that are likely to appear. Consider the standard Hamiltonian $H(\boldsymbol{q},\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}^2)$ ($\boldsymbol{q}$ and $\boldsymbol{p}$ are d-dimensional vectors) and try to implement $C(\boldsymbol{q},\boldsymbol{p}) = \boldsymbol{q}\cdot\boldsymbol{p}$ as a constraint. Following the above instructions we get $\boldsymbol{P}(\boldsymbol{q},\dot{\boldsymbol{q}},\lambda) = m(\dot{\boldsymbol{q}} - \lambda\boldsymbol{q})$ and λ is determined as $\Lambda(\boldsymbol{q},\dot{\boldsymbol{q}}) = m\frac{\boldsymbol{q}\cdot\dot{\boldsymbol{q}}}{\boldsymbol{q}^2}$. A substitution of both determinations of $\boldsymbol{p}$ and λ into the extended Lagrangian yields $L(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}m\dot{\boldsymbol{q}}\mathbb{M}\dot{\boldsymbol{q}} - V(\boldsymbol{q}^2)$, where $\mathbb{M}$ is the matrix $\mathbb{M}_{ij} = \delta_{ij} - \frac{q_i q_j}{\boldsymbol{q}^2}$. This Lagrangian is singular because the Hessian matrix with respect to the velocities is (up to a multiplicative constant) identical to $\mathbb{M}$, i.e. to a projector transverse to $\boldsymbol{q}$. Thus $L(\boldsymbol{q},\dot{\boldsymbol{q}})$ may potentially describe a gauge theory.

The Lagrangian momenta are defined as $\hat{\boldsymbol{p}} = \frac{\partial L}{\partial\dot{\boldsymbol{q}}} = \mathbb{M}\dot{\boldsymbol{q}}$, which indeed implies the constraint $\boldsymbol{q}\cdot\boldsymbol{p} \simeq 0$ because $\mathbb{M}\boldsymbol{q} = 0$ identically. The canonical Hamiltonian is just $\frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}^2)$. So the dynamics in phase space is given by the Dirac Hamiltonian $H_D(\boldsymbol{q},\boldsymbol{p}) := H(\boldsymbol{q},\boldsymbol{p}) + \eta\boldsymbol{q}\cdot\boldsymbol{p}$, as expected. The problem in this example is that we must require stabilization of the now primary constraint $\boldsymbol{q}\cdot\boldsymbol{p} \simeq 0$. We get, as the secondary constraint, $\frac{\boldsymbol{p}^2}{2m} - \boldsymbol{q}^2 V'(\boldsymbol{q}^2) \simeq 0$. For a general potential V this gives a new condition which in its turn must be stabilized again, and so on. We can easily end up with incompatibility. Nothing of this kind happens if we choose the constraint as one of the constants of motion of the theory.

4.2. Enforcing a constant of motion as a constraint

Let us work with the same example as in the previous section, i.e. $H(\boldsymbol{q},\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}^2)$, but now in $\mathbb{R}^3$, and with $C(\boldsymbol{q},\boldsymbol{p}) = \epsilon^{3jk}q^j p^k$. The latter is nothing but one of the conserved angular momenta. With this we get $P^l(\boldsymbol{q},\dot{\boldsymbol{q}},\lambda) = m(\dot{q}^l - \lambda\epsilon^{3jl}q^j)$. Insertion of $\boldsymbol{P}(\boldsymbol{q},\dot{\boldsymbol{q}},\lambda)$ into the constraint C determines

$$\Lambda(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{\epsilon^{3jk}q^j\dot{q}^k}{\alpha}, \tag{27}$$

with $\alpha := (q^1)^2 + (q^2)^2$. Upon evaluation and elimination of p and λ, we obtain from the extended Lagrangian L_{e} the new Lagrangian

$$L(\boldsymbol{q}\dot{\boldsymbol{q}}) = \frac{m}{2}\left[\dot{\boldsymbol{q}}^2 - \frac{(\epsilon^{3jk}q^j\dot{q}^k)^2}{\alpha}\right] - V(\boldsymbol{q}^2) = \frac{m}{2}\dot{\boldsymbol{q}}\mathbb{M}\dot{\boldsymbol{q}} - V(\boldsymbol{q}^2), \tag{28}$$

with the projector

$$\mathbb{M}^{nk} = \delta^{nk} - \frac{\epsilon^{3mn}\epsilon^{3jk}q^m q^j}{\alpha}. \tag{29}$$

It is easy to check that the projector $\mathbb{M}$ has $v^k := \epsilon^{3jk}q^j$ as the null vector.

Now we work with the Lagrangian (28). The Lagrangian momenta are $\hat{\boldsymbol{p}} = \partial L/\partial\dot{\boldsymbol{q}} = \mathbb{M}\dot{\boldsymbol{q}}$. The canonical Hamiltonian becomes again $\frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}^2)$ but there is the primary constraint $\epsilon^{3jk}q^j p^k$ which is now derived from the definition of the canonical momenta and the use of the null vector for $\mathbb{M}$. Thus, the Dirac Hamiltonian is $H_D(\boldsymbol{q},\boldsymbol{p}) := H(\boldsymbol{q},\boldsymbol{p}) + \eta\epsilon^{3jk}q^j p^k$. Stabilization of this constraint is trivial and there are no secondary constraints in phase space. In agreement with this fact, one can check that the Lagrangian (28) does not yield constraints in tangent (i.e. configuration-velocity) space.

One can identify the gauge transformation for L as $\delta_L q^i = \epsilon(t)\{q^i, C\}|_{p\to P, \lambda\to\Lambda} = -\epsilon(t)\epsilon^{3ij}q^j$. It is more instructive to read it by taking cylindrical coordinates z, ρ, θ; then, $\delta_L z = 0, \delta_L \rho = 0, \delta_L \theta = \epsilon$. In these coordinates the Lagrangian (28) is

$$L = \tfrac{1}{2}m(\dot{z}^2 + \dot{\rho}^2) - V(z^2 + \rho^2). \tag{30}$$

Now the gauge symmetry becomes obvious because there is no dependence on the angular variable in the Lagrangian. Indeed the variable θ is purely gauge. The original, non-gauge, Lagrangian was $L_{\rm ng} = \frac{1}{2}m(\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\theta}^2) - V(z^2 + \rho^2)$, so we see that the whole procedure boils down to getting rid of the piece $\rho^2\dot{\theta}^2$. This term was invariant under rigid translations for the variable θ, that is, rigid rotations around the z-axis. The disappearance of this term makes these rotations a gauge symmetry.

An illuminating consideration can be drawn from this example. At first sight it could come as a surprise that the implementation of the constraint, which requires the vanishing of the 'angular momentum' along the z-axis, allows for motions whose projection to the x–y plane has arbitrary dependence in the variable θ. The correct way of looking at it is the other way around: in promoting the constant of motion $\epsilon^{3jk}q^j p^k$ to be a constraint, we are also promoting it from being a rigid symmetry generator to a gauge generator; consequently, the rotations around the z-axis are promoted to gauge transformations. In group theoretical terms, the implementation of $\epsilon^{3jk}q^j p^k$ as a constraint has the consequence that a subgroup of the original rigid symmetry $SO(3)$ gets gauged, precisely that of the rotations around the z-axis.

4.3. Relativistic massive spinless particle

Consider the Lagrangian (spacetime indices will be mostly suppressed) $L_{\rm ng} = \frac{1}{2}m\dot{x}^2$ in Minkowski spacetime with $\eta_{\mu\nu} = {\rm diag}(1, -1, \ldots, -1)$, and the rest mass m. Its associated Hamiltonian is $H = \frac{1}{2m}p^2$. All the momenta are constants of motion, so we can try to implement them as constraints. We then get the extended Lagrangian

$$L_{\rm e} = p\dot{x} - \frac{1}{2m}p^2 - \lambda(p - a), \tag{31}$$

where in component notation $\lambda(p - a) \equiv \lambda^\mu(p_\mu - a_\mu)$, and a_μ is a constant 4-vector. Elimination of the momenta yields the Lagrangian

$$L_\lambda = \tfrac{1}{2}m(\dot{x} - \lambda)^2 + \lambda a, \tag{32}$$

which has the gauge symmetry $\delta_\lambda x^\mu = \epsilon^\mu(\tau), \delta_\lambda \lambda^\mu = \dot{\epsilon}^\mu(\tau)$, with $\epsilon^\mu(\tau)$ being arbitrary infinitesimal functions of the evolution parameter. If we further eliminate the variables λ^μ, which have by now acquired the status of auxiliary variables, we end up with the Lagrangian

$$L = a\dot{x} - \frac{1}{2m}a^2. \tag{33}$$

The last term is an irrelevant constant. Note that the EOM for L are void: every trajectory is a solution of the EOM. This conclusion should not be surprising because all translational symmetries in the Minkowski target space have been gauged, which results in making any trajectory acceptable as a solution of the EOM. We have simply introduced too much gauge freedom.

Instead of trying to gauge the rigid translations in the target space, we could have decided to gauge the rigid translations along the world line, that is, the rigid translations in the evolution parameter. Its associated symmetry in phase space is $\delta_\epsilon x = \epsilon\dot{x}, \delta_\epsilon p = 0$, with ϵ being an infinitesimal constant and the generator is the constant of motion $\frac{1}{2}p^2$. Let us fix the value of this constant of motion so that $p^2 = m^2$ and require this relation to become a constraint. This value $p^2 = m^2$ selects trajectories with unit velocity in Minkowski spacetime, $\dot{x}^2 = 1$, but after enforcing this constant of motion as a constraint, a very different setting emerges, as we will see. For later convenience we consider the rescaled constant of motion $C = \frac{1}{2m}(p^2 - m^2)$. In this case,

$$L_{\mathrm{e}} = p\dot{x} - \frac{1}{2m}p^2 - \frac{\lambda}{2m}(p^2 - m^2), \tag{34}$$

and the elimination of the momenta gives

$$L_\lambda = \frac{m}{2(1+\lambda)}\dot{x}^2 + \frac{1}{2}\lambda m, \tag{35}$$

which indeed has the gauge symmetry $\delta_\lambda x = \epsilon(\tau)\frac{\dot{x}}{1+\lambda}$, $\delta_\lambda \lambda = \dot{\epsilon}(\tau)$, obtained under the rules given in section 3. Addition to L_λ of an irrelevant constant $m/2$ (which does not affect the dynamics), and a redefinition $\lambda \to \lambda - 1$ allows us to write the modified Lagrangian (for which we keep the same notation) as

$$L_\lambda = \frac{m}{2\lambda}\dot{x}^2 + \frac{1}{2}\lambda m, \tag{36}$$

with gauge transformations $\delta_\lambda x = \epsilon(\tau)\frac{\dot{x}}{\lambda}$, $\delta_\lambda \lambda = \dot{\epsilon}(\tau)$. Defining as a new arbitrary function $\xi = \frac{\epsilon}{\lambda}$, the infinitesimal gauge transformations read

$$\delta_\lambda x = \xi \dot{x}, \qquad \delta_\lambda \lambda = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi\lambda), \tag{37}$$

which show directly that x is a scalar and λ a scalar density under the reparametrization $\tau \mapsto \tau - \xi$. The reader may rightly recognize in L_λ the familiar Wheeler–Polyakov's Lagrangian [17, 18]

$$L_{\mathrm{WP}} = -\tfrac{1}{2}(e^{-1}(\tau)\dot{x}^\mu(\tau)\dot{x}_\mu(\tau) + e(\tau)m^2), \tag{38}$$

with $\lambda = -me$. The auxiliary variable $e(\tau)$ is an *einbein* (i.e. square-root of the world-line metric) and τ is the world-line parameter ('label time'). It can be easily checked that the corresponding action for L_λ is invariant under finite reparametrizations of the label time, $\tau \mapsto \tau' = f(\tau)$, which, in the *active* view of reparametrization invariance, read

$$x^\mu(\tau) \mapsto x'^\mu(\tau) = x^\mu(f^{-1}(\tau)), \qquad \lambda(\tau) \mapsto \lambda'(\tau) = \left(\frac{\mathrm{d}f^{-1}(\tau)}{\mathrm{d}\tau}\right)\lambda(f^{-1}(\tau)). \tag{39}$$

Here $f(\tau)$ is an arbitrary monotonically increasing function of τ. It is easy to check that the finite transformations (39) can be obtained from the infinitesimal transformations (37) if we set $f(\tau) = \tau - \xi$ and successively iterate.

The next step is to get rid of the variable λ via the scheme presented in section 3.4. The final Lagrangian L becomes $L = m\sqrt{\dot{x}^2}$, which coincides with the usual square root world-line Lagrangian for the relativistic particle. The corresponding action is well known to be invariant under reparametrizations of the label time (i.e. under the first transformation in (39)). We have thus succeeded in making the original theory invariant under reparametrizations (or diffeomorphisms). As a bonus we have recovered the (on mass-shell) equivalence between L_{WP} and the square root world-line Lagrangian.

5. The gauge principle in relativistic field theory

5.1. The minimal setting

Let us apply our results to a non-Abelian field theory. For definiteness we will consider an N-component complex scalar field that transforms under the fundamental representation of the

$SU(N)$ group. The corresponding (non-gauge) Lagrangian density for the free fields is given by

$$\mathcal{L}_{\rm ng} = \eta^{\mu\nu}(\partial_\mu \phi^*) \cdot (\partial_\nu \phi) - m^2 \phi^* \cdot \phi. \tag{40}$$

This Lagrangian has clearly $SU(N)$ rigid symmetry

$$\delta\phi = \mathrm{i}\epsilon^a T_a \phi, \qquad \delta\phi^* = -\mathrm{i}\epsilon^a \phi^* T_a, \tag{41}$$

(note henceforth that the action of the Hermitian matrix T_a in $\phi^* T_a$ undergoes a transposition with respect to the action of T_a in $T_a\phi$) with ϵ^a being infinitesimal constants and T_a the Hermitian $(N \times N)$ matrices spanning a basis of the Lie algebra of $SU(N)$, $[T_a, T_b] = \mathrm{i} f_{ab}^c T_c$. To make the rigid transformation gauge, we proceed along the methods outlined in sections 2 and 3. Let us first move the description in phase space. The Lagrangian definition of the momenta is

$$\pi = \partial_0 \phi^*, \qquad \pi^* = \partial_0 \phi, \tag{42}$$

and the Hamiltonian density becomes

$$\mathcal{H} = \pi^* \cdot \pi + (\nabla_i \phi^*) \cdot (\nabla_i \phi) + m^2 \phi^* \cdot \phi. \tag{43}$$

The constants of motion which generate the rigid $SU(N)$ symmetry are obtained as coefficients of the infinitesimal constants ϵ^a in the space integration of the time component of the conserved current, which is computed by standard Noether methods (see, e.g., [9]). We get

$$j^0(x) = \mathrm{i}\epsilon^a [\pi(x) \cdot T_a \phi(x) - \phi^*(x) T_a \cdot \pi^*(x)]. \tag{44}$$

The generator $G = \epsilon^a G_a := \int \mathrm{d}^3\boldsymbol{x}\, j^0(x)$ indeed generates (41) together with

$$\delta\pi = -\mathrm{i}\epsilon^a \pi T_a, \qquad \delta\pi^* = \mathrm{i}\epsilon^a \phi^* T_a. \tag{45}$$

These transformations are in full agreement with the definition of the Lagrangian momenta (42). The algebra of the generators

$$G_a = \mathrm{i} \int \mathrm{d}^3\boldsymbol{x} [\pi(x) \cdot T_a \phi(x) - \phi^*(x) T_a \cdot \pi^*(x)] \tag{46}$$

is $\{G_a, G_b\} = -f_{ab}^c G_c$. The opposite sign in front of the structure constant f_{ab}^c is a direct consequence of the conventional choice $[T_a, T_b] = \mathrm{i} f_{ab}^c T_c$. The contact with our results from section 2 can be established by taking $c_{ab}^c = -f_{ab}^c$.

The extended Lagrangian now takes the form

$$\begin{aligned} L_{\rm e} = \int \mathrm{d}^3\boldsymbol{x} \mathcal{L}_{\rm e} = \int \mathrm{d}^3\boldsymbol{x} (&\pi \cdot \dot\phi + \dot\phi^* \cdot \pi^* - \pi^* \cdot \pi - (\nabla_i \phi^*) \cdot (\nabla_i \phi) \\ &- m^2 \phi^* \cdot \phi - \mathrm{i}\lambda^a (\pi \cdot T_a \phi - \phi^* T_a \cdot \pi^*)). \end{aligned} \tag{47}$$

The gauge transformations for $\mathcal{L}_{\rm e}$ are given by (41) and (45), but with ϵ^a now being an arbitrary infinitesimal function of time, together with the analog of (8):

$$\delta\lambda^a(x) := \partial_0 \epsilon^a(t) - f_{bc}^a \epsilon^b(t) \lambda^c(x) = (D_0 \epsilon(t))^a. \tag{48}$$

Next we proceed as in section 3 to construct the Lagrangian $\mathcal{L}_\lambda$. We obtain, after some simple computations

$$\mathcal{L}_\lambda = (D_0\phi)^* (D_0 \phi) - (\nabla_i \phi^*) \cdot (\nabla_i \phi) - m^2 \phi^* \cdot \phi, \tag{49}$$

with

$$D_0 \phi := \partial_0 \phi - \mathrm{i}\lambda^a T_a \phi, \qquad (D_0\phi)^* := \partial_0 \phi^* + \mathrm{i}\lambda^a \phi^* T_a \tag{50}$$

being the usual gauge covariant derivatives with the standard covariance condition $\delta(D_0\phi) = \mathrm{i}\epsilon^a(x) T_a D_0 \phi$.

5.2. Finishing the job

We have succeeded with $\mathcal{L}_\lambda$ in implementing gauge invariance in a restricted form. In fact, we have implemented it in the most minimal way, by adding as many new fields—the old Lagrange multipliers—as dimensions of the original rigid group we have gauged, and by restricting the infinitesimal parameters $\epsilon^a(t)$ of the gauge transformation to be only functions of time, albeit arbitrary. On the other hand, the above implementation was so minimal that we have lost a big chunk of the Poincaré invariance along the way. Looking at the structure of the term $D_0\phi$, it is clear that if Poincaré transformations are to be implemented in their entirety, the fields λ^a are nothing else than the time components A_0^a of vector fields A_μ^a, as $\partial_0\phi$ are time components of the vector fields $\partial_\mu\phi$. Now we can in a single stroke restore full Poincaré invariance and also let the gauge parameters to have arbitrary dependence on all the spacetime coordinates. We just need to mimic what has been done for the time coordinate for all the space coordinates. In this way, gauge invariance is trivially preserved and we recover Poincaré invariance. Then the term $\partial_i\phi$ in the Lagrangian (49) must be modified to $D_i\phi := \partial_i\phi - \mathrm{i}A_i^a T_a\phi$ and similarly for $\partial_i\phi^*$. The gauge transformations for the gauge fields will be the generalization of (48), namely $\delta A_\mu^a(x) := \partial_\mu\epsilon^a(x) - f_{bc}^a\epsilon^b(x)A_\mu^c(x)$. All in all we end up with the well-known Lagrangian

$$\mathcal{L} = \eta^{\mu\nu}(D_\mu\phi^*)\cdot(D_\nu\phi) - m^2\phi^*\cdot\phi, \tag{51}$$

which is the Lagrangian for the minimal coupling of the complex scalar fields with the gauge field.

5.3. The direct way: De Donder–Weyl formalism

The way of finishing the job in the previous subsection leaves us with the uneasiness of having done it with some artifice. The problem is that the standard canonical formalism destroys the explicit Lorentz invariance and the procedure in subsection 5.1 ends up with truly destroying Lorentz invariance, which then must be restored 'by hand', as done in subsection 5.2. Fortunately there is a better way. De Donder–Weyl formalism [16], which preserves manifest Lorentz invariance in phase space, is a more suited tool to do the job. Let us go back to the Lagrangian (40) and define the Lorentz 4-component momenta (polymomenta) by

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial_\mu\phi} = \partial^\mu\phi^*, \qquad \pi^{*\mu} = \frac{\partial\mathcal{L}}{\partial_\mu\phi^*} = \partial^\mu\phi. \tag{52}$$

The Hamiltonian, defined in the De Donder–Weyl formalism (DWF) through $\pi^\mu\cdot\partial_\mu\phi + \pi^{*\mu}\cdot\partial_\mu\phi^* - \mathcal{L}_{\mathrm{ng}}$, becomes

$$\mathcal{H}_{\mathrm{DW}} = \pi^\mu\cdot\pi^{*\nu}\eta_{\mu\nu} + m^2\phi^*\cdot\phi. \tag{53}$$

To write the extended Lagrangian we will use all four components of the $SU(N)$ conserved currents, $j_a^\mu = \mathrm{i}(\pi^\mu T_a\cdot\phi - \phi^* T_a\cdot\pi^{*\mu})$. This is the natural way in DWF to maintain a manifest Lorentz invariance[6]. The associated multipliers A_μ^a are then Lorentz 4-vectors. De Donder–Weyl's extended Lagrangian can then be written as

$$\mathcal{L}_{\mathrm{e}} = (\pi^\mu)\cdot\partial_\mu\phi + (\pi^\mu)^*\cdot\partial_\mu\phi^* - \mathcal{H}_{\mathrm{DW}} - \mathrm{i}A_\mu^a(\pi^\mu\cdot T_a\phi - \phi^* T_a\cdot\pi^{*\mu}). \tag{54}$$

Finally, applying the methods introduced in section 3, we can successively construct Lagrangians $\mathcal{L}_\lambda$ and $\mathcal{L}$. By calling the latter as $\mathcal{L}_{\mathrm{DW}}$, we obtain

$$\mathcal{L}_{\mathrm{DW}} = \eta^{\mu\nu}(D_\mu\phi^*)\cdot(D_\nu\phi) - m^2\phi^*\cdot\phi, \tag{55}$$

[6] To strengthen this point note that the currents associated with gauge symmetries must always vanish on shell, except for possible divergences of arbitrary antisymmetric tensors. In the De Donder–Weyl formalism all the components of the gauge currents are constraints in the new phase space.

with the covariant derivatives as defined above: $D_\mu\phi := \partial_\mu\phi - \mathrm{i}A^a_\mu T_a\phi$, etc. By finding $\mathcal{L}_{\mathrm{DW}}$ we have gained a new conceptual access to gauge field theories in flat spacetime.

From here, the rest is straightforward. One can find the curvature $[D_\mu, D_\nu]$ which transforms under the adjoint representation of the gauge group and allows for a simple construction of a gauge-invariant Lagrangian with kinetic terms for the Yang–Mills gauge fields—and a bonus of new couplings in the non-Abelian case. With covariant derivatives and curvatures at one's disposal, one can analogously formulate other gauge field theories such as Chern–Simons gauge theory or BF gauge theory [19]. Non-local gauge invariants like Wilson loops or effective gluon masses [20] are also at hand.

We have worked out the case of N-component complex scalar field transforming under the $SU(N)$ fundamental representation but we could have done the same, e.g. for the real-valued field multiplet in the $SO(N)$ fundamental representation and for the spinorial case (e.g. for Dirac or Rarita–Schwinger fields). Note that the Abelian case is recovered just as a particular case, as it should be.

6. World sheet general covariance: the Nambu–Goto closed string

As another relevant example, we consider the non-gauge Lagrangian

$$\mathcal{L}_{\mathrm{ng}} = \frac{T}{2} h^{ab}\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} := \frac{T}{2} h^{ab}\partial_a x \partial_b x, \tag{56}$$

with the world-sheet metric $h_{ab} = \mathrm{diag}(1, -1)$ and the target-space (or background) metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$. T is the string tension. For simplicity we will in the following work with natural units where $T = 1$. The target-space functions $x^\mu(\tau, \sigma)$ describe the spacetime embedding of the world sheet. In the following, we will suppress the target-space indices. Our aim now is to gauge the world-sheet rigid translational symmetry

$$\delta_\epsilon x = \epsilon^a \partial_a x. \tag{57}$$

To prevent any conflicting issue concerning the 'spatial' (σ) boundary conditions we will deal exclusively in this section with the closed string. Following section 5.3, the De Donder–Weyl polymomenta are $p^a = \frac{\partial\mathcal{L}}{\partial(\partial_a x)} = h^{ab}\partial_b x$, and the corresponding De Donder–Weyl Hamiltonian becomes

$$\mathcal{H}_{\mathrm{DW}} = p^a\partial_a x - \mathcal{L}_{\mathrm{ng}} = \tfrac{1}{2} h_{ab} p^a p^b. \tag{58}$$

The Noether conserved current associated with symmetry (57) is found by ordinary methods to be

$$J^a = \epsilon^b \left(p^a h_{bc} p^c - \tfrac{1}{2}\delta^a_b p^d h_{dc} p^c \right). \tag{59}$$

In addition to $\delta_\epsilon x$, we need also to know $\delta_\epsilon p^a$. To compute it we resort momentarily to the standard canonical formalism and proceed as follows. The world sheet τ-component of the current is

$$J^0 = \epsilon^b \left(p^0 h_{bc} p^c - \tfrac{1}{2}\delta^0_b p^d h_{dc} p^c \right), \tag{60}$$

where $p^1 = h^{11}\partial_1 x = -x'$, so J^0 has the explicit form

$$J^0 = \epsilon^0 \left((p^0)^2 - \frac{1}{2}[(p^0)^2 - (x')^2] \right) + \epsilon^1 (p^0 x') = \frac{\epsilon^0}{2}[(p^0)^2 + (x')^2] + \epsilon^1 (p^0 x'). \tag{61}$$

From this expression, the transformations of p^0 mediated by the corresponding Noether charge read

$$\delta_\epsilon p^0 = \int \mathrm{d}\sigma' \left\{ p^0(\tau, \sigma), J^0(\tau, \sigma') \right\} = \partial_1(\epsilon^0 x' + \epsilon^1 p^0). \tag{62}$$

In deriving (62) we have allowed for ϵ^a to be an arbitrary infinitesimal world-sheet function to prepare the formalism for the gauge transformations we want to implement.

By rewriting $\delta_\epsilon p^0$ with the help of De Donder–Weyls' polymomenta we get $\delta_\epsilon p^0 = \partial_1(\epsilon^1 p^0 - \epsilon^0 p^1)$. Since in the DWF all polymomenta play the same role, we infer that the general transformation law for p^a is

$$\delta_\epsilon p^a = \partial_b(\epsilon^b p^a - \epsilon^a p^b). \tag{63}$$

This should be coupled together with transformations (57) which in terms of the De Donder–Weyl variables read

$$\delta_\epsilon x = \epsilon^a h_{ab} p^b. \tag{64}$$

This last transformation also naturally follows from our definition of variations δ_ϵ (cf equation (4)), namely

$$\delta_\epsilon x = \int \mathrm{d}\sigma' \left\{x(\tau,\sigma), J^0(\tau,\sigma')\right\} = \epsilon^a h_{ab} p^b, \tag{65}$$

as it, of course, should.

Next, in order to proceed with our program, we define the extended Lagrangian with Lagrange multipliers A_{ab}. By remembering that target-space indices are suppressed, we obtain

$$\begin{aligned}\mathcal{L}_\mathrm{e} &= p^a\partial_a x - \mathcal{H}_\mathrm{DW} - A_{ab}\big(p^a p^b - \tfrac{1}{2}h^{ab}h_{dc}p^d p^c\big)\\ &= p^a\partial_a x - \tfrac{1}{2}h_{ab}p^a p^b - \tfrac{1}{2}B_{ab}p^a p^b,\end{aligned} \tag{66}$$

where $\frac{1}{2}B_{ab} := A_{ab} - \frac{1}{2}h_{ab}A_{cd}h^{cd}$ is symmetric and traceless. This shows that although we initially had three free Lagrange multipliers (A_{ab} is symmetric) we end up with only two, because of the particular structure of the current J^a and the dimensionality of the world sheet.

Note the important fact that the new EOM for $\mathcal{L}_\mathrm{e}$ imply $\partial_a x = (h_{ab} + B_{ab})p^b$, and therefore expression (64), originated from (57) before the implementation of the Lagrange multipliers, needs to be reformulated to $\delta_\epsilon x = \epsilon^a(h_{ab} + B_{ab})p^b$. In turn this means (cf equation (65)) that the conserved current needs to be reformulated. It should be noticed that a redefinition of currents has not been requisite in the previously discussed systems (apart from the relativistic particle in section 4.3) because the Noether currents—coming from rigid (target-space) symmetries—do not change when the constraints are imposed. In contrast, here we deal with currents that come from rigid world-sheet symmetries and these are influenced when we change $\mathcal{L}_\mathrm{ng}$ to $\mathcal{L}_\mathrm{e}$. Clearly, the same scenario occurs also for the relativistic particle discussed in section 4.3, but there the change from $\delta_\epsilon x = \epsilon p/m$ to $\delta_\epsilon x = \epsilon p(1+\lambda)/m$ can be assimilated into a redefinition of ϵ without any extra consequences. This is not the case here (see our discussion later). It is also important to observe that $\delta_\epsilon p^a$ as defined by equation (63) is not altered because the metric tensor does not appear in expression (63) and one can check that the changes in the current are exactly absorbed, as regards the computation of $\delta_\epsilon p^0$, with the redefinition of the relation between $\partial_a x$ and p^b, already mentioned. The above-outlined redefinition of the conserved current is just the first step in an iteration process, with the aim of consistency, in which we know that at every step the current will be quadratic in the momenta. Thus this process will result in a final extended Lagrangian of the general form

$$\mathcal{L}_\mathrm{f} = p^a\partial_a x - \tfrac{1}{2}C_{ab}p^a p^b, \tag{67}$$

where C_{ab}, which we take symmetric, contains all the information about the Lagrange multipliers. Seen in retrospect, (66) should be interpreted as the first-order expansion of C_{ab} around the world sheet Minkowski metric, so that $C_{ab} = h_{ab} + B_{ab}$, with the coefficients B_{ab} now taken infinitesimal. Once this observation is taken into account, we note that the tracelessness condition for B_{ab} amounts to the condition $\det C_{ab} = -1$ for this $C_{ab} = h_{ab}+B_{ab}$.

Thus $\det C_{ab} = -1$ is valid at first order around h_{ab}. Repeated iterations of the infinitesimal change $h_{ab} \mapsto h_{ab} + B_{ab}$ will be expected to preserve this condition (cf subsection 6.2). Thus we end up with the result that the final extended Lagrangian is supplemented by the condition

$$\det C_{ab} = -1. \tag{68}$$

The consequences of (68) will be explored later on, in the next subsection.

If our inputs are correct, the Lagrangian (67) should exhibit gauge freedom under the transformations (with ϵ^a arbitrary infinitesimal functions),

$$\delta_\epsilon x = \epsilon^\alpha C_{ab} p^b, \qquad \delta_\epsilon p^a = \partial_b(\epsilon^b p^a - \epsilon^a p^b), \tag{69}$$

and a certain (so far unknown) transformation $\delta_\epsilon C_{ab}$. This means that $\delta_\epsilon C_{ab}$ should be such that together with (69) it should leave the Lagrangian $\mathcal{L}_{\mathrm{f}}$ quasi-invariant, i.e. with $\delta_\epsilon \mathcal{L}_{\mathrm{f}}$ being a divergence. Let us now prove the consistency of our scheme by providing the explicit form for $\delta_\epsilon C_{ab}$. To this end we first write

$$\delta_\epsilon \mathcal{L}_{\mathrm{f}} = (\delta_\epsilon p^a)\partial_a x + p^a \partial_a(\delta_\epsilon x) - C_{ab}(\delta_\epsilon p^a) p^b - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b, \tag{70}$$

and note that the first term is already a divergence because

$$(\delta_\epsilon p^a)\partial_a x = \partial_b(\epsilon^b p^a - \epsilon^a p^b)\partial_a x = \partial_b[(\epsilon^b p^a - \epsilon^a p^b)\partial_a x]. \tag{71}$$

Thus ((div.) stands for divergences),

$$\begin{aligned}
\delta_\epsilon \mathcal{L}_{\mathrm{f}} &= (\text{div.}) + p^a \partial_a(\epsilon^c C_{cb} p^b) - C_{ab}(\delta_\epsilon p^a) p^b - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b \\
&= (\text{div.}) + p^a \partial_a(\epsilon^c C_{cb}) p^b + p^a \epsilon^c C_{cb}(\partial_a p^b) - C_{ab}(\partial_c(\epsilon^c p^a - \epsilon^a p^c)) p^b - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b \\
&= (\text{div.}) + p^a \partial_a(\epsilon^c C_{cb}) p^b + p^a \epsilon^c C_{cb}(\partial_a p^b) + (\epsilon^c p^a - \epsilon^a p^c)\partial_c(C_{ab} p^b) - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b \\
&= (\text{div.}) + p^a \partial_a(\epsilon^c C_{cb}) p^b + p^a \epsilon^c C_{cb}(\partial_a p^b) + (\epsilon^c p^a - \epsilon^a p^c)(\partial_c C_{ab}) p^b \\
&\quad + (\epsilon^c p^a - \epsilon^a p^c) C_{ab}(\partial_c p^b) - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b.
\end{aligned} \tag{72}$$

Consider the next to last term in (72), i.e. $(\epsilon^c p^a - \epsilon^a p^c) C_{ab}(\partial_c p^b)$. The second piece cancels another term in (72), whereas the first piece can be written as

$$\epsilon^c p^a C_{ab}(\partial_c p^b) = \tfrac{1}{2}\epsilon^c C_{ab}\partial_c(p^a p^b) = (\text{div.}) - \tfrac{1}{2}\partial_c(\epsilon^c C_{ab}) p^a p^b. \tag{73}$$

All in all we end up with

$$\begin{aligned}
\delta_\epsilon \mathcal{L}_{\mathrm{f}} &= (\text{div.}) + \tfrac{1}{2} p^a(\partial_a(\epsilon^c C_{cb}) + \partial_b(\epsilon^c C_{ca})) p^b + (\epsilon^c p^a - \epsilon^a p^c)(\partial_c C_{ab}) p^b \\
&\quad - \tfrac{1}{2}\partial_c(\epsilon^c C_{ab}) p^a p^b - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b \\
&= (\text{div.}) + \tfrac{1}{2} p^a(\partial_a(\epsilon^c C_{cb}) + \partial_b(\epsilon^c C_{ca})) p^b + p^a \epsilon^c(\partial_c C_{ab}) p^b \\
&\quad - \tfrac{1}{2} p^a(\epsilon^c(\partial_a C_{cb}) + \epsilon^c(\partial_b C_{ca})) p^b - \tfrac{1}{2}\partial_c(\epsilon^c C_{ab}) p^a p^b - \tfrac{1}{2}(\delta_\epsilon C_{ab}) p^a p^b,
\end{aligned} \tag{74}$$

which implies that under the transformation

$$\delta_\epsilon C_{ab} = \epsilon^c \partial_c C_{ab} + C_{cb}\partial_a \epsilon^c + C_{ac}\partial_b \epsilon^c - C_{ab}\partial_c \epsilon^c, \tag{75}$$

the Lagrangian $\mathcal{L}_{\mathrm{f}}$ is indeed quasi-invariant. Note that this solution (75) for the transformations rules of C_{ab} is unique. Equation (75) is the Lie derivative of a covariant tensor density (0, 2) of weight -1 along ϵ, i.e. $\delta_\epsilon C_{ab} = \pounds_\epsilon C_{ab}$. Its inverse matrix, which we denote as C^{ac}, will then be a contravariant tensor density (2, 0) of weight +1, which then transforms according to

$$\delta_\epsilon C^{ab} = \epsilon^c \partial_c C^{ab} - C^{cb}\partial_c \epsilon^a - C^{ac}\partial_c \epsilon^b + C^{ab}\partial_c \epsilon^c = \pounds_\epsilon C^{ab}. \tag{76}$$

Result (75) is very good news because the elimination of the momenta from their own EOM in (66) produces the Lagrangian $\mathcal{L}_\lambda$—which in this context is more reasonable to denote as $\mathcal{L}_{\mathrm{C}}$ (and similarly substitute δ_λ by δ_{C})—which reads

$$\mathcal{L}_{\mathrm{C}}(x, \partial_a x, C_{bc}) = \mathcal{L}_{\mathrm{f}}(x, \partial_a x, P^c(x, \partial_a x, C_{de},), C_{de}) = \tfrac{1}{2} C^{ab}\partial_a x \partial_b x. \tag{77}$$

The latter is a scalar density under the transformations (57) and (76), indeed $\delta_C \mathcal{L}_C = \partial_a(\epsilon^a \mathcal{L}_C)$. Because transformations (57) and (76) are respectively Lie derivatives for scalars and for tensor densities, they—similarly as in the general relativity [9]—express diffeomorphism invariance (or general covariance) of the theory.

6.1. *The condition* $\det C = -1$

The Lagrangian $\mathcal{L}_C$ is not the end of the story because the auxiliary variables C_{ab} satisfy the additional condition $\det C_{ab} = -1$. First note that this condition is compatible with the gauge symmetry because $\det C_{ab}$ behaves as a scalar under the gauge transformation (75):

$$\delta_\epsilon \det C = \epsilon^a \partial_a (\det C). \tag{78}$$

As a by-product, we see that by requiring the extra constraint $\det C_{ab} = -1$, the gauge freedom stays intact.

In practice, one may consider two ways to implement the condition $\det C_{ab} = -1$ into $\mathcal{L}_C$. One possible procedure is to introduce new gauge freedom by defining $C_{ab} = \frac{1}{\sqrt{-g}} g_{ab}$, with g_{ab} an arbitrary symmetric tensor in the world sheet of signature $\{+, -\}$ and $g := \det g_{ab}$ (note that $\det(\frac{1}{\sqrt{-g}} g_{ab}) = -1$ and $C^{ab} = \sqrt{-g} g^{ab}$). The new gauge freedom is Weyl invariance, $g_{ab} \mapsto \Lambda(\tau, \sigma) g_{ab}$. This new gauge freedom compensates for the fact that g_{ab} has three components whereas C_{ab} had only two. The result is the familiar nonlinear σ model Lagrangian [21, 22][7] for bosonic string theory,

$$\mathcal{L}_\sigma = \tfrac{1}{2}\sqrt{-g} g^{ab} \partial_a x \partial_a x. \tag{79}$$

It is well known that at the classical level one can eliminate g_{ab}, which are auxiliary variables in (79), by plugging their own EOM into (79). The result is the Nambu–Goto Lagrangian. However, from the quantum-mechanical view, the issue is more delicate. Instead of eliminating g_{ab} via its EOM, one should perform a Feynman path integral, and use the standard Fadeev–Popov procedure to deal with the local symmetries and gauge fixing. When this is done correctly [18], one finds that there is a conformal anomaly unless the target-space dimension is $D = 26$. But even in 26 dimensions, it is not yet clear whether off mass-shell fluctuations of the Nambu–Goto and the nonlinear σ-model actions contribute in the same way, say into string partition function. As we are interested here only in classical level description we will not pursue this point further.

The second procedure consists in enforcing $\det C = -1$ with a Lagrange multiplier. One modifies the Lagrangian (77) so that the new Lagrangian is

$$\tilde{\mathcal{L}}_C = \tfrac{1}{2} C^{ab} \partial_a x \partial_b x + \lambda(t - 1), \tag{80}$$

where $t := \sqrt{-\det C_{ab}}$ (the square root is introduced for a technical convenience). Since the first term in (80) is already a scalar density, the transformation properties of the multiplier λ must also be those of a scalar density, i.e. $\delta_C \lambda = \partial_a(\epsilon^a \lambda)$. Using the fact that C_{ab} have become auxiliary variables for (80), we obtain from their own EOM that $C_{ab} = \frac{1}{\lambda t} \partial_a x \partial_b x$, and therefore t is determined as

$$t = \frac{1}{\sqrt{\lambda}} (-\det \partial_a x \partial_b x)^{\frac{1}{4}}. \tag{81}$$

Plugging this result into (80), we get

$$\tilde{\mathcal{L}}_C = 2\sqrt{\lambda} (-\det \partial_a x \partial_b x)^{\frac{1}{4}} - \lambda. \tag{82}$$

[7] Actually the name string (nonlinear σ) model Lagrangian is a little unfair, since the Lagrangian (79) was discovered independently by several researchers. But in our view it is a bit clumsy to talk about the Brink–Di Vecchia–Howe–Deser–Zumino–Polyakov Lagrangian.

Now the multiplier λ has turned an auxiliary variable. Its EOM determines $\lambda = (-\det \partial_a x \partial_b x)^{\frac{1}{2}}$. Substitution of this result into (82) yields the Nambu–Goto Lagrangian

$$\mathcal{L}_{\mathrm{NG}} = (-\det \partial_a x \partial_b x)^{\frac{1}{2}}. \tag{83}$$

This again reconfirms the fact that on mass-shell $\mathcal{L}_\sigma \cong \mathcal{L}_{\mathrm{NG}}$.

6.2. Further considerations

There is a strong parallelism between our way of obtaining the world-sheet general covariance and the approach [23] to general relativity out of the requirement of self-consistency of the coupling of the energy–momentum tensor of an initially Minkowskian theory to a massless spin-2 field. The presence of the coupling term produces changes in the energy–momentum tensor which in its turn redefine the coupling term, making it nonlinear in the spin-2 field. An interaction procedure is set to work and the final result is the appearance of the metric tensor field and general covariance. In our case, the Lagrange multipliers B_{ab} play the role of the spin-2 field. A self-consistency requirement also appears because the conserved current for world sheet translation invariance has changed due to the presence of the new term with the multipliers. In fact in the DWF, we enforce all the components of the current to become constraints, and thus the Lagrange multipliers B_{ab} are in fact coupled to the energy–momentum tensor. The difference is that in our case, due to the particular structure of the current, we end up with a density tensor field C_{ab} of weight -1 that must satisfy $\det C_{ac} = -1$.

Let us elaborate a bit more on the requirement $\det C_{ab} = -1$. This condition is crucial for our purposes. In fact we have found the fulfillment of this condition for configurations of C_{ab} around the flat spacetime metric and we have checked that the extension of this result to any configuration is fully compatible with gauge freedom. We could also argue that since we have found only two degrees of freedom—those of traceless symmetric B_{ab}—around the flat spacetime metric, to preserve this number we must accept that the components of C_{ab} are constrained by a condition of the type $f(C_{ab}) =$ constant. If we make the reasonable assumption that this condition is geometrical—since the Lagrangian (83) already is—we conclude that it should be a scalar under diffeomorphisms. But the only scalar we can construct out of the components of the tensor density C_{ab} is just its determinant, and to fix its value we need only to consider the configurations around h_{ab}.

It is remarkable that as a way to perform the covariant quantization of the bosonic string, Kato and Ogawa [24] used essentially the Lagrangian (80) as a Lagrangian equivalent to (79). On the other hand, Siegel [25], see also [26], used the extended Lagrangian (67) with the specific requirement $\det C_{ab} = -1$. In our approach (67) and (80) are consequences of gauging the world sheet rigid translational symmetry of the Lagrangian (56).

Finally let us stress that the dimensionality of the world sheet plays a crucial role in our derivation of the Nambu–Goto Lagrangian (83) through gauging the rigid world sheet translational symmetry (57). It is only when the world sheet is two dimensional that the Lagrange multipliers are constrained so as to satisfy an additional condition which eventually leads to the requirement $\det C_{ab} = -1$.

7. Conclusions

Let us summarize our findings. Our starting point is a non-gauge theory, defined by a regular Lagrangian L_{ng}. We assume that in the phase space formulation such a theory has a Lie algebra of time-independent constants of motion. Next we enforce these constants of motion as first class constraints by adding them to the Hamiltonian with a set of Lagrange multipliers. Then

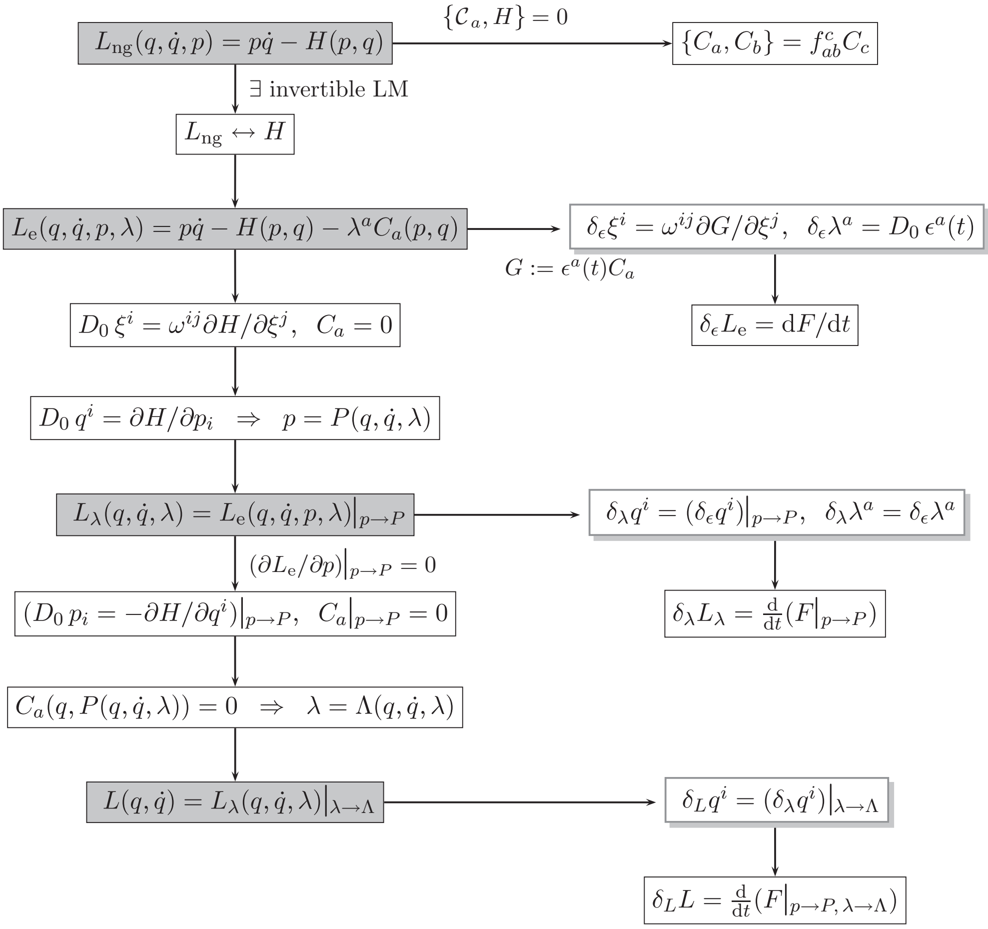

Figure 1. The sequence diagram summarizing the basic logical steps leading from L_{ng} to L. The abbreviation LM stands for Legendre map while F denotes some phase-space function which is linear in ϵ and its derivatives.

we perform the inverse Legendre transformation to end up with a new (extended) Lagrangian L_{e} whose configuration space now includes the Lagrange multipliers as new variables. We then observe that this new theory has gauge symmetries and that the gauge group is generated by the constraints, as expected. We also observe that in general the new variables are auxiliary and that they can be further eliminated from the formalism by plugging into the new Lagrangian their determination through their own equations of motion. This yields the final gauge-invariant Lagrangian L. This last step may be problematic with regard to quantization because the final theory will in general be of non-polynomial nature. Another option is to enlarge the theory with the addition of new gauge-invariant terms that make these auxiliary variables dynamical. The passage from L_{ng} to L is schematically illustrated in the sequence diagram in figure 1.

In the special case of relativistic field theories we have noticed that our program is best carried out if the canonical setting is taken along the lines of the De Donder–Weyl approach. Such formalism is particularly suitable because it keeps manifest the Lorentz invariance from the very beginning. The simplicity with which this gauging procedure can be performed within this formalism is remarkable.

We have illustrated the DWF by applying it to the case of the N-component complex scalar field transforming under the $SU(N)$ fundamental representation but we could have done the same, e.g., for the real-valued field multiplet in the $SO(N)$ fundamental representation, for the spinorial case, etc. It should be, nevertheless, noted that the role of the DWF is purely instrumental, and that once the Lagrangian for the gauge theory has been obtained (see, for instance, equation (55)), one can proceed either with the Lagrangian or with standard canonical methods, without having to rely again on the DWF.

As another relevant example we have derived the Nambu–Goto Lagrangian for the closed bosonic string by gauging the world sheet rigid translational symmetry of a non-gauge Lagrangian. Our strategy has again relied on the DWF and it entailed an iteration procedure very close to the approach to Einstein's general theory of gravitation [23] in which a consistency argument on the coupling of a massless spin 2 field with the total energy–momentum tensor (including matter fields) yields ultimately the Einstein–Hilbert action. It should be, however, stressed that because in our reasonings the dimensionality of the world sheet has played a crucial role, it is not yet clear if a similar iterative procedure can be applied, e.g. to relativistic Dirac–Nambu–Goto membranes (or p-branes).

The above-considered examples clearly indicate that the gauge principle, i.e. the gauging of a rigid group of symmetries, can be alternatively recast in the language of constrained systems with the gauge fields appearing first as Lagrange multipliers for the enforcement of the constants of motion as constraints. The rationale of the procedure is based on the fact that rigid symmetries are generated by constants of motion, whereas gauge symmetries by first class constraints. Thus to gauge a group of rigid symmetries is tantamount to enforce the generating constants of motion as constraints. Note also that the role of the gauge fields as multipliers is temporary, because after the implementation of the inverse Legendre transformation they typically become auxiliary variables. Finally, when the Lagrangian is modified with new gauge-invariant additions to provide for kinetic terms for the gauge fields, they become dynamical variables on their own.

We notice also that the constraints C_a, directly originated from the former constants of motion of the nongauged theory, are primary constraints, but that does not mean that our framework is limited to this kind of constraints and cannot give rise to secondary constraints. In contrast, the examples provided in section 5 show that, due to the presence of the kinetic terms for the gauge fields—which are the former Lagrangian multipliers—in the final Lagrangian, secondary constraints may arise, as it is indeed the case for the Yang–Mills gauge theories.

With the benefit of hindsight, we observe that this route of enforcing constants of motion as constraints could have been taken from the very beginning as an alternative way to the gauge principle, because at the time when the Yang–Mills theory was formulated, the foundations and development of the theory of constrained systems were already in place.

We believe that the presented formulation can also be conveniently applied in the 't Hooft program [27] where the extended Lagrangians (3) formulated with the help of constants of motion have played a pivotal role in the construction of emergent dynamical systems [28, 29]. This issue would deserve further investigation.

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